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LETTER TO THE EDITOR

Nonlinear dynamics of a continuous spring–block model of earthquake faults

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Abstract. The continuous one-dimensional Burridge–Knopoff model is generalized by introducing plastic creep in addition to rigid sliding. The resulting equations, for an order parameter (sliding rate) and a control parameter (driving force), exhibit a velocity-strengthening and a velocity-softening instability. In the former regime, reminiscent of self-organized criticality in continuum systems, anomalous diffusion is described by a nonlinear diffusion equation. The latter regime, characteristic of deterministic chaos, is described by a time-dependent Ginzburg–Landau equation. Implications of the model with respect to earthquake predictability are discussed.

The dynamics of non-trivial spring–block models of earthquakes has regained attention since Bak and Tang [1] argued that crustal faults may exhibit self-organized criticality (SOC). It was shown that if stick–slip dynamics of a slowly driven system with threshold dynamics is computed by a cellular-automaton algorithm the system self-tunes into a statistical steady state with power-law correlations in time and in space [2–4]. These correlations compare favourably to experimentally observed power-law correlations for earthquakes (for example, to the Gutenberg–Richter law for the frequency distribution of energy release, or to the Omori law for the distribution of aftershocks [5]).

While the SOC models are based on *strain softening* (i.e. the threshold is defined by a critical force in the force–strain characteristic), a complementary approach to obtaining power-law scaling has been based on spring–block models involving *velocity softening*. These models (both discrete and continuous versions), whose simplest form is known as the Burridge–Knopoff (BK) model [6], generate stick–slip instabilities through a negative velocity sensitivity of the dynamic fault friction. Power-law scaling is reproduced [7, 8] by these deterministically chaotic models even in the absence of an explicit stochastic element in the continuous differential equation: randomness is introduced only via the initial conditions. However, recent investigations have shown that the solutions of BK models based on a simple velocity-softening dynamic friction law eventually settle to periodic cycles of large events [9].

We present a generalization of the continuous, one-dimensional uniform BK model. The necessity of this generalization is motivated by physical arguments. (i) Since velocity softening originates from fault aging by plastic accommodation (creep) of the fault interface, it is not consistent to consider softening in terms of a phenomenological friction law

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without accounting for the time scale of aging [10]. In remedying this shortcoming we shall introduce time-dependent (memory) friction. (ii) A significant fraction of the imposed tectonic drift is accommodated by smooth creep deformation (aseismic slip), while only some part (sometimes less than 1%) can be attributed to the abrupt strain release during earthquakes (seismic slip). Understanding this observed seismic slip deficit [11] and the concomitant intermittent behaviour (clusters of events separated by quiescent periods) necessitates the consideration of creep *and* rigid sliding as concurring modes of accommodation of the tectonic drift. (iii) An apparent dichotomy persists in the literature regarding the concepts of strain softening (as envisioned by SOC models) and velocity softening (BK model and its derivates). Since the dynamics of these classes of models is qualitatively different, any progress may have important implications as to earthquake predictability. In addressing this question we hope to contribute to the recent debate [12–14] of continuum realizations of SOC.

The standard BK model of earthquake faults describes the dynamics of displacements u of a slowly driven spring–block chain of masses m in the presence of a nonlinear dynamic friction Φ that generates a velocity-softening instability. The masses (of characteristic extension ξ) are longitudinally coupled by coil springs of stiffness k_{l} and they are transversally coupled to the driving interface by leaf springs of stiffness k_{t}. Here we propose to account for fault creep via the introduction of an internal variable. Specifically, under the imposed driving the fault responds by a combination of rigid translation u_{s} (sliding) and plastic displacement u_{p} (irreversible deformation by creep of some boundary layer of the fault). In the continuum limit the corresponding equations of motion become (primes denote spatial derivatives and dots time derivatives)

$$m\ddot{u}_{\mathrm{s}} = F - \Phi(\dot{u}_{\mathrm{s}} + \dot{u}_{\mathrm{p}}) \tag{1}$$

$$\dot{F} = k_{\mathrm{l}}\xi^2\dot{u}_{\mathrm{s}}'' - k_{\mathrm{t}}(\dot{u}_{\mathrm{s}} - \bar{v}) - \frac{F - F_{\mathrm{y}}}{\tau_{\mathrm{p}}}. \tag{2}$$

The first equation describes balance of forces (Newton's law) and the second the time evolution of the driving shear force F. The external loading rate is $k_{\mathrm{t}}\bar{v}$, where $\bar{v}$ is the tectonic drift velocity. The first term on the right-hand side of (2) arises from longitudinal compression or tension, the second from elastic shear, and the last gives the force relaxation due to plastic deformation under the assumption that the plastic displacement rate depends linearly on the driving force,

$$\dot{u}_{\mathrm{p}} = \frac{F - F_{\mathrm{y}}}{k_{\mathrm{t}}\tau_{\mathrm{p}}} \qquad \text{for } F \geqslant F_{\mathrm{y}}. \tag{3}$$

The critical force corresponding to plastic yielding is denoted by F_{y} and τ_{p} is the characteristic time of plastic relaxation of the shear forces (aging). In what follows we neglect F_{y}, without loss of generality.

In analogy to critical phenomena equations (1) and (2) may be interpreted in terms of the coupled dynamics of a *control parameter* F (or the plastic displacement rate $\dot{u}_{\mathrm{p}}$) and an *order parameter* $\dot{u}_{\mathrm{s}}$ (the sliding rate). This compares with the proposed feedback mechanism of Gil and Sornette [14]. Moreover, such a description is reminiscent of previous analyses of laboratory friction data [15, 16] where a set of two coupled differential equations for the friction stress, a parameter that characterizes the evolving state of the surface, and a constitutive equation have been proposed. For the friction law of [17] and in the absence of inhomogeneities ($\dot{u}_{\mathrm{s}}'' = 0$) these equations may be rewritten in a form similar to ours: the equation for the driving force (2) becomes identical, whereas the equation corresponding to (1) contains additional nonlinearities [18].

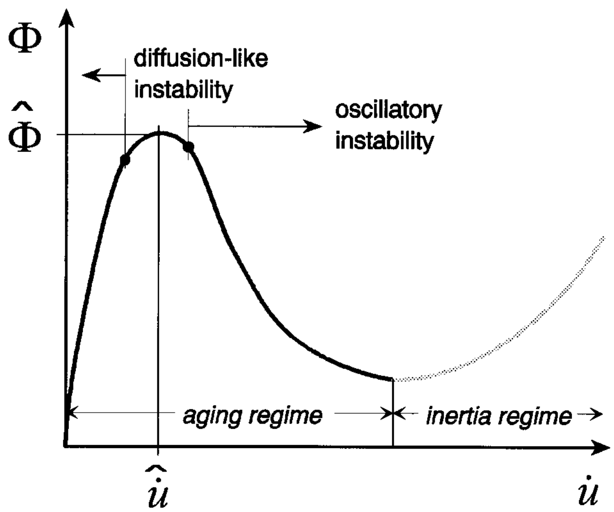

Figure 1. The nonlinear friction force assumed in the generalized BK model. Note the velocity-strengthening and velocity-softening region, separated by the friction-force maximum $\widehat{\Phi}$ at $\widehat{\dot{u}}$. Typical positions of the diffusion-like and the oscillatory instabilities are shown. The aging and the inertia regimes are presented for completeness.

A schematic diagram of the friction force Φ we will be considering is shown in figure 1: for small $\dot{u} = \dot{u}_s + \dot{u}_p$ fault creep is fast enough (time scale τ_p) to maintain optimum surfaces of contact which results in a positive velocity sensitivity of Φ. Velocity softening results from incomplete fault accommodation occurring in an intermediate velocity range where the displacement rates are compatible with the rate of aging. The velocity sensitivity turns positive again at those high velocities where aging is negligible as compared to inertia. It is important to note that Φ defines the dynamic friction in the steady state, while velocity changes are associated with transient effects as a consequence of memory friction [18].

Time and space are rendered dimensionless by defining $\tilde{t} = t/\tau_p$ and $\tilde{x} = (k_t/k_l)^{1/2}x/\xi$, while the dimensionless sliding rate is defined by $e = \dot{u}_s\tau_p/l_p$, the dimensionless shear force by $f = F/(k_t l_p)$, and the dimensionless friction force by $\phi = \Phi/(k_t l_p)$. By this scaling we have introduced a characteristic length scale $l_p = \tau_p \bar{v}$. In terms of the scaled fields, equations (1) and (2) become

$$\epsilon^2 \dot{e} = f - \phi[\bar{v}(e + f)] \tag{4}$$

$$\dot{f} = e'' - e - f + 1. \tag{5}$$

Besides the set of constants defining the friction force ϕ, equations (4) and (5) depend on two parameters: the imposed drift velocity $\bar{v}$ defining the 'working point' of the fault, and $\epsilon = (m/k_t)^{1/2}/\tau_p$ which is the ratio of the natural frequency of transversal oscillations of individual blocks to the plastic relaxation time. It is tempting to identify the $\epsilon \to 0$ ($\tau_p \to \infty$) limit as the standard, homogeneous BK model. However, inspection of the scaling relations and equation (3) shows that the BK limit corresponds to $\tau_p \to \infty$ with $\dot{u}_p \to 0$, namely creep is suppressed while the friction force remains constant. The natural limit of equations (4) and (5) is to let ϵ become small keeping the plastic displacement rate $\dot{u}_p$ fixed, since the plastic relaxation time τ_p and the characteristic length scale l_p go to infinity at the same rate (for fixed $\bar{v}$). The physical interpretation of this limit is that an increase of the plastic relaxation time is associated with an increase of the driving force F since force relaxation by creep becomes less efficient†. Consequently, the limit that gives the standard BK model is not physically reasonable in our framework since the shear force and the plastic relaxation time are interrelated (cf equation 3) [18].

In the case of small ϵ where friction dominates over inertial effects one can eliminate adiabatically the fast variable e (the order parameter $\dot{u}_s$). This yields the evolution of the slow variable f (control parameter F)

$$\dot{f} = [D(f)f']' - \partial_f V(f) \tag{6}$$

$$D = \frac{1}{\bar{v}}\partial_f \zeta(f) - 1 \qquad V = \frac{1}{\bar{v}}\int_0^f \mathrm{d}z\,\zeta(z) - f \tag{7}$$

where ζ is the inverse function of ϕ, $\zeta(f) \equiv \phi^{-1}(f)$. Since the friction force is not injective, ζ is defined only in the part of the aging regime where the slope of the friction force is positive (cf figure 1). Equation (7) shows that for $\partial_f\zeta/\bar{v} < 1$ (the friction-force slope greater than unity) the diffusion coefficient becomes negative: $D < 0$ (*uphill diffusion*). As the friction-force maximum $\widehat{\Phi}$ is approached, however, D becomes singular ($\partial_f\zeta \to \infty$), so that close to $\widehat{\Phi}$ the adiabatic approximation is expected to break down as the time scale associated with diffusive relaxation of the control parameter vanishes.

It is apparent that this *diffusion-like instability*, which is not present in the original BK model, contains many of the features previously attributed to SOC. It is characterized by an uphill diffusion and, in the adiabatic approximation, the diffusion constant becomes singular at the friction-force maximum $\widehat{\Phi}$. Either criterion has been identified as characteristic of SOC in continuum systems [12–14]. To make this point clear we consider figure 2 which gives a schematic representation of the force dependence of (a) the coefficient D governing the nonlinear diffusion and (b) the potential V introduced in equations (6) and (7). On the one hand, the uphill-diffusion regime ($D < 0$) mimics self-tuning, inasmuch as the velocity field tends to fragmentize. Tentatively, we attribute this to microfissuration. On the other hand, the singularity in D at $\widehat{\Phi}$ defines an unsticking threshold at which the force is distributed instantaneously. Since the threshold is separated from the self-tuning regime by a gap characterized by stable diffusion ($D > 0$), some element of randomness is necessary to overcome the potential barrier and to trigger an event. Noise is supposed to be provided by material inhomogeneities along real faults.

The nature of instabilities in equations (4) and (5) is elucidated by performing a linear stability analysis of the uniform, steady-state solution ($e_0 = 1 - \phi(\bar{v}), f_0 = \phi(\bar{v})$) to infinitesimal perturbations $[\delta e(0), \delta f(0)]\exp(\omega t + \mathrm{i}kx)$. In doing so we have to distinguish between 'working points' $\bar{v}$ falling into the velocity strengthening regime at small velocities, and those within the velocity-softening regime at intermediate velocities (the inertia regime at high velocities is linearly stable, but unstable with respect to finite-amplitude perturbations, cf figure 1). The roots of the characteristic polynomial are

$$\omega_\pm = \frac{1}{\epsilon^2}(\mu \pm \{\mu^2 - \epsilon^2[1 + (1 - \phi_0^{(1)}k^2]\}^{1/2}) \tag{8}$$

$$\mu = -(\epsilon^2 + \phi_0^{(1)})/2 \qquad \phi_0^{(n)} \equiv \bar{v}^n \partial_{\bar{v}}^n \phi(\bar{v}). \tag{9}$$

We identify two kinds of instabilities depending on the sign of μ. For $\mu \geqslant 0$ a transition occurs from a stable focus to an unstable focus via a *Hopf bifurcation*. The corresponding

† Clearly, the (theoretical) shear strength of the bulk material that constitutes the fault poses an upper bound to the shear force. Here we consider the limit $\epsilon \ll 1$.

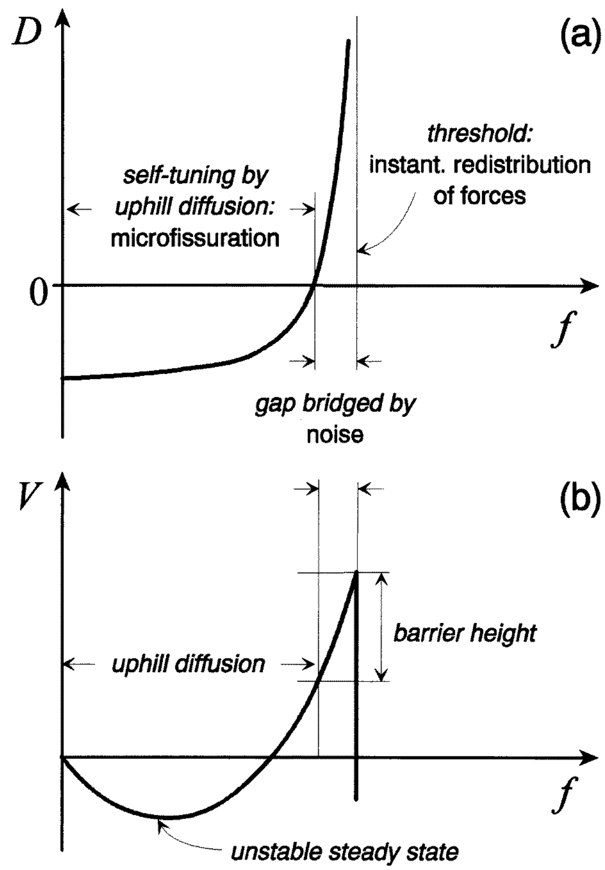

Figure 2. An illustration of the force dependence of (a) the diffusion-like coefficient D and (b) the potential V introduced in equations (6) and (7). For a discussion, see text.

purely imaginary eigenvalues are

$$\omega_{\pm} = \pm \frac{\mathrm{i}}{\epsilon}[1 + (1+\epsilon^2)k^2]^{1/2} \qquad \text{for } \mu = 0. \tag{10}$$

This velocity-softening instability has been identified previously [7] in the standard BK model, except that the introduction of plastic deformation shifts the instability to a finite negative value of the steady-state velocity sensitivity ($\phi_0^{(1)} < -\epsilon^2$), cf figure 1. This is a consequence of the stabilizing influence of the plastic deformation. Furthermore, by an appropriate physical interpretation of the relaxation time τ_{p} [18], the Hopf bifurcation criterion can be mapped onto the criterion for stick–slip instabilities proposed by Ruina [15].

The dynamics of the system close to the oscillatory instability (in the post-bifurcation regime) and the nature of the Hopf bifurcation are investigated in terms of a time-dependent Ginzburg–Landau (TDGL) equation for a complex order parameter A [19]. This equation is derived by introducing new variables $g = e + f - 1$ and $h = f - f_0$, and by expanding the friction force ϕ about the steady-state solution ($g = 0$) up to third order in g. Close to the bifurcation point slow modes which govern the dynamics of the system are introduced by scaling space and time according to $X = x/\epsilon$ and $T = t/\epsilon^2$, where ϵ is assumed to be small. The reduced evolution equations derive in a standard way [18, 19] by performing an expansion of the solution vector in terms of the complex eigenvectors of the linear evolution operator followed by an expansion in the small parameter μ that measures the distance to the bifurcation point. This results in a TDGL equation which, expressed in a standard form, is

$$\partial_T A = -\mathrm{i}\partial_X^2 A + A - (1 + \mathrm{i}c)|A|^2 A \tag{11}$$

and which depends on a single dimensionless parameter $c = 2(\phi_0^{(2)})^2/(3\epsilon\phi_0^{(3)}) > 0$ and c real [18]. As the amplitude A scales like $1/\sqrt{\mu}$ it follows that the Hopf bifurcation is supercritical (continuous transition), in agreement with experimental results on dry-friction dynamics [20]. The constant c is positive ($\phi_0^{(3)} > 0$) because we may assume that the inflection point in ϕ occurs at velocities larger than those that define the bifurcation point (which is close to the maximum of ϕ).

It is well known that equation (11) possesses a spatially uniform oscillating in time solution $A(T) = \exp(-\mathrm{i}cT)$ that describes the motion on the limit cycle enclosing the bifurcation point in phase space. This uniform motion is unstable with respect to finite wavelength perturbations (sideband instability or modulational instability [21, 22]). The associated eigenvalues read [18]

$$\Omega_\pm = -1 \pm (1 + 2cK^2 - K^4)^{1/2}. \tag{12}$$

For $c > 0$† the uniform state is unstable (Ω_+ positive) in the range of wavenumbers $K = \epsilon k$ with $0 < |K| < \sqrt{2c}$ with maximum growth occurring at $|K| = \sqrt{c}$.

Lorenz-type amplitude equations have been analysed to study the period-doubling transition to chaos in the TDGL equation which is associated with the existence of a strange attractor [22]. For the purposes of this work, it suffices to comment that the transition to chaos depends sensitively on the friction force law (in particular, on the sign of the third derivative).

In addition to the oscillatory instability, the original equations (4) and (5) exhibit a velocity-strengthening instability occurring for $\mu < 0$. For $\phi_0^{(1)} > 1$ the second term under the square root in equation (8) becomes positive, and an instability associated with small wavelengths, $k^2 \geqslant (\phi_0^{(1)} - 1)^{-1}$, appears. This instability has no analogue in the standard BK model and it is closely related to the memory friction introduced in our generalized model. Since $\phi_0^{(1)} > 1$ corresponds to $D < 0$ in equation (7), we identify it as the *diffusion-like instability* that appeared in the analysis of the adiabatic elimination of the fast variable.

The identification of two distinct regimes in the generalized BK model presented herein may have significant implications for the predictability of earthquakes. When the fault operates in the velocity-strengthening regime at low velocities (tectonic drift velocity $\bar{v} < \widehat{u}$), one has aseismic slip most of the time while the diffusion-like instability coupled to internal noise due to fault inhomogeneities gives self-tuning of the system to a state that has some of the properties previously attributed to SOC states. This dynamics is infinite-dimensional and hence prediction becomes impossible because small perturbations may make parts of the system enter the intermediate velocity-softening regime leading to large excursions and major events. This interpretation is in line with current ideas that earthquakes are unpredictable [23]. Once the system is, however, in the velocity-softening regime, the evolution becomes chaotic according to a low-dimensional deterministic dynamics. Therefore, in principle, monitoring and time-series analysis of a small number of modes will allow short-term forecasting (on a time scale governed by the largest Lyapunov exponent) of the fault evolution during the aftershock dynamics. As we assume the initial working

† This condition may be considered a modified Newell criterion (cf [21, 22]).

point to be $\bar{v} < \widehat{u}$, the fault will ultimately return to the velocity-strengthening regime which corresponds to the beginning of another seismic cycle.

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References

[1] Bak P and Tang C 1989 *J. Geophys. Res.* **94** 15 635
[2] Bak P, Tang C and Wiesenfeld K 1987 *Phys. Rev. Lett.* **59** 381
Bak P, Tang C and Wiesenfeld K 1988 *Phys. Rev.* A **38** 364
[3] Olami Z and Feder H J S 1992 *Phys. Rev. Lett.* **68** 1244
Christensen K and Olami Z 1992 *Phys. Rev.* A **46** 1829
[4] Chen K, Bak P and Obukhov S P 1991 *Phys. Rev.* A **43** 625
[5] Ito K and Matsuzaki M 1990 *J. Geophys. Res.* **95** 6853
[6] Burridge R and Knopoff L 1967 *Bull. Seismol. Soc. Am.* **57** 3411
[7] Carlson J M and Langer J S 1989 *Phys. Rev. Lett.* **62** 2632
Carlson J M and Langer J S 1989 *Phys. Rev.* A **40** 6470
Carlson J M, Langer J S, Shaw B E and Tang C 1991 *Phys. Rev.* A **44** 884
Myers C R and Langer J S 1993 *Phys. Rev.* E **47** 3048
Carlson J M, Langer J S and Shaw B E 1994 *Rev. Mod. Phys.* **66** 657
[8] de Sousa Vieira M, Vasconcelos G L and Nagel S R 1993 *Phys. Rev.* E **47** R2221
Vasconcelos G L, de Sousa Vieira M and Nagel S R 1992 *Physica* A **191** 69
[9] Rice J R 1993 *J. Geophys. Res.* **98** 9885
Xu H-J and Knopoff L 1994 *Phys. Rev.* E **50** 3577
[10] Zaiser M and Hähner P 1997 *Phys. Status Solidi* b **199** 267
[11] DeMets C 1997 *Nature* **386** 549
[12] Carlson J M, Chayes J T, Grannan E R and Swindle G H 1990 *Phys. Rev. Lett.* **65** 2547
[13] Hwa T and Kardar M 1989 *Phys. Rev. Lett.* **62** 1813
[14] Gil L and Sornette D 1996 *Phys. Rev. Lett.* **76** 3991
[15] Ruina A 1983 *J. Geophys. Res.* **88** 10 359
[16] Marone C J, Scholz C H and Bilham R 1991 *J. Geophys. Res.* **96** 8441
[17] Rice J R and Gu J-C 1983 *Pure Appl. Geophys.* **121** 187
[18] Hähner P and Drossinos Y 1997 submitted
[19] Haken H 1975 *Z. Phys.* B **21** 105
Kuramoto Y and Tsuzuki T 1975 *Prog. Theor. Phys.* **54** 687
[20] Heslot F, Baumberger T, Perrin B, Caroli B and Caroli C 1994 *Phys. Rev.* E **49** 4973
[21] Benjamin T B and Feir J E 1967 *J. Fluid Mech.* **27** 417
Newell A C and Whitehead J A 1969 *J. Fluid Mech.* **38** 279
[22] Moon H T 1993 *Rev. Mod. Phys.* **65** 1535
[23] Geller R J, Jackson D D, Kagan Y Y and Mulargia F 1997 *Science* **275** 1616